UV/IR Mixing for
Noncommutative Complex Scalar Field Theory, II
(Interaction with Gauge Fields)

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Abstract

We consider noncommutative analogs of scalar electrodynamics and $\mathcal{N} = 2$ $D = 4$ SUSY Yang-Mills theory. We show that one-loop renormalizability of noncommutative scalar electrodynamics requires the scalar potential to be an anticommutator squared. This form of the scalar potential differs from the one expected from the point of view of noncommutative gauge theories with extended SUSY containing a square of commutator. We show that fermion contributions restore the commutator in the scalar potential. This provides one-loop renormalizability of noncommutative $\mathcal{N} = 2$ SUSY gauge theory. We demonstrate a presence of non-integrable IR singularities in noncommutative scalar electrodynamics for general coupling constants. We find that for a special ratio of coupling constants these IR singularities vanish. Also we show that IR poles are absent in noncommutative $\mathcal{N} = 2$ SUSY gauge theory.

1 Introduction

Recently, there is a renovation of the interest in noncommutative quantum field theories (or field theories on noncommutative space-time [1, 2]). As emphasized in [3], the important question is whether or not the noncommutative quantum field theory is well-defined. Note that one of earlier motivations to consider noncommutative field theories is a hope that it would be possible to avoid quantum field theory divergencies [4, 5, 2, 6, 7, 8]. One could think that a theory on a noncommutative space is renormalizable only if the corresponding commutative theory is renormalizable. Results on one-loop renormalizability of noncommutative gauge theory [9] and two-loop renormalizability of noncommutative scalar ϕ_4^4 theory [10] as well as general considerations [11, 12] support this belief. But this expectation is not true for more complicated models. In [13] a noncommutative quantum field theory of a complex scalar field with the interaction $\lambda^2(a\phi^* \star \phi \star \phi^* \star \phi + b\phi^* \star \phi^* \star \phi \star \phi)$ has been considered. Here λ is a coupling constant. It has been shown that this model is renormalisable at one-loop in two special cases: $a = b$ and $b = 0$.

In this paper we analyze an interaction of the noncommutative complex scalar field with the noncommutative $U(1)$ gauge field (noncommutative scalar electrodynamics). We argue that one-loop renormalizability requires the scalar potential to be an anticommutator squared. However, $\mathcal{N} = 2$ supersymmetric Yang-Mills theory [14, 15] contains a complex scalar field with the potential which is equal to the square of the commutator. At the first sight it seems that the above fact contradicts to expected renormalisability of noncommutative $\mathcal{N} = 2$ SUSY Yang-Mills theory [16, 17]. Nevertheless, it turns out that fermions remove this discrepancy. Namely, fermion contributions restore a commutator in the scalar potential. Thus our analysis confirms one-loop renormalisability of noncommutative $\mathcal{N} = 2$ SUSY Yang-Mills theory.

Note that UV renormalizability does not guarantee that the theory is well-defined. There is a mixing of the UV and IR divergencies [18, 10]. UV/IR mixing depends on the model [13, 16]. The $U(1)$ noncommutative gauge theory does not exhibit a mixing of the UV and the IR dynamics [19]. For further developments in perturbative study of noncommutative field theories see [21]-[30], for non-perturbative aspects see [31]-[34]. In this paper we show that one can remove IR divergencies in noncommutative scalar electrodynamics in the case of a special relation between two coupling constants. Also we show that IR poles are absent in noncommutative $\mathcal{N} = 2$ SUSY gauge theory.

The paper is organized as follows. In Section 2 we analyze noncommutative scalar electrodynamics. In Section 3 we consider noncommutative $\mathcal{N} = 2$ SUSY Yang-Mills theory (NC SYM) and prove its one-loop renormalizability. In Section 4 we analyze infrared behaviour of noncommutative scalar electrodynamics and $\mathcal{N} = 2$ NC SYM. We show that the latter theory is well defined meanwhile the first one requires a special relation between two coupling constants.

2 Scalar Electrodynamics

Let us start with a consideration of noncommutative scalar electrodynamics in Euclidean space $\mathbb{R}^4$. The classical action is given by

$$S = \int d^4x \left(-\tfrac{1}{4}F_{\mu\nu} \star F^{\mu\nu} + (\mathcal{D}_\mu\phi^*) \star (\mathcal{D}_\mu\phi) + V[\phi^*, \phi]\right),$$
$$V[\phi^*, \phi] = \lambda^2(a\phi^* \star \phi \star \phi^* \star \phi + b\phi^* \star \phi^* \star \phi \star \phi), \tag{1}$$

where $\star$ is a Moyal product $(f \star g)(x) = \exp(-i\xi\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu)f(x) \otimes g(x)$ with $\theta^{\mu\nu}$ being a real constant skew-symmetric nondegenerate matrix and ξ being a deformation parameter. The covariant derivative is defined by $\mathcal{D}_\mu\phi = \partial_\mu\phi - ig[A_\mu, \phi]_\star$, where $[A_\mu, \phi]_\star = A_\mu \star \phi - \phi \star A_\mu$. g and λ are coupling constants and a and b are fixed real numbers. It has been shown in [13], that the pure complex scalar field theory is one-loop renormalizable only if $a = b$ or $b = 0$. The purpose of the present analysis is to find the analogous restrictions on a and b in the case of scalar electrodynamics.

The action (1) is invariant under the following gauge transformations

$$\phi \mapsto U \star \phi \star U^\dagger, \quad \phi^* \mapsto U \star \phi^* \star U^\dagger, \quad A_\mu \mapsto U \star A_\mu \star U^\dagger - \partial_\mu U \star U^\dagger, \tag{2}$$

where U is an element of the noncommutative $U(1)$ group [17]. Note that since our fields are in adjoint representation we could consider the theory with two real scalar fields instead of one complex.

The Feynman rules for the theory (1) are presented in Table 1. Solid lines denote the scalar fields, "in" arrows stand for the field ϕ and "out" arrows stand for the field ϕ^*.
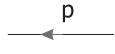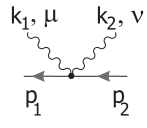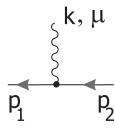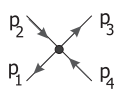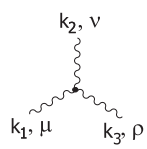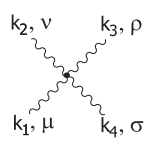
	$D_{\mu\nu}(k) = -\frac{1}{k^2}\left[\delta_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right]$
	$D(p) = \frac{1}{p^2}$
	$4g^2\delta^{\mu\nu}[\cos(k_1 \wedge p_1 + k_2 \wedge p_2) - \cos(p_1 \wedge p_2)\cos(k_1 \wedge k_2)]$
	$2ig(p_1 - p_2)_\mu \sin(p_1 \wedge p_2)$
	$-4\lambda^2[a\cos(p_1 \wedge p_2 + p_3 \wedge p_4) + b\cos(p_1 \wedge p_3)\cos(p_2 \wedge p_4)]$
	$-2ig\sin(k_1 \wedge k_2)[(k_1 - k_2)_\rho\delta_{\mu\nu} + (k_2 - k_3)_\mu\delta_{\nu\rho} + (k_3 - k_1)_\nu\delta_{\mu\rho}]$
	$4g^2[(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho})\sin(k_1 \wedge k_2)\sin(k_3 \wedge k_4)$ $+(\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho})\sin(k_1 \wedge k_3)\sin(k_2 \wedge k_4)$ $+(\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\rho}\delta_{\nu\sigma})\sin(k_1 \wedge k_4)\sin(k_2 \wedge k_3)]$

Table 1: Feynman rules for scalar electrodynamics.

We do not specify the Feynman rules for ghosts, since they do not contribute to one-loop graphs with the external matter lines.

All calculations are performed in Landau gauge $\alpha = 0$. This is a convenient choice, since in this gauge a great number of graphs do not have divergent parts. One can prove that the theory is gauge invariant on quantum level (cf. [9, 16]), so our results and conclusions are valid for an arbitrary value of α.

The above mentioned restriction on a and b can come from one-loop corrections to the 4-scalar and 2-scalar-2-gluon vertices. First we consider one-loop corrections to the 4-point scalar vertex. The graphs that have non zero divergent parts in Landau gauge are presented in Figure 1. Using the dimensional regularization ($D = 4 - 2\epsilon$) we find that the sum of
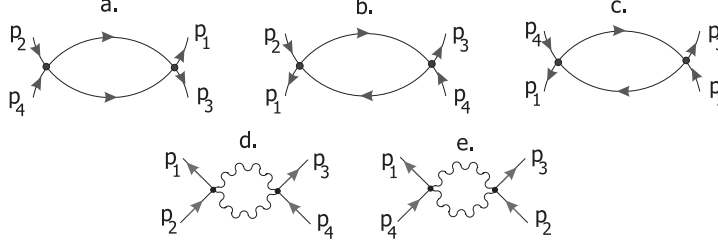

Figure 1: One-loop corrections to the 4-point scalar vertex.

divergent parts of these graphs is equal to

$$\frac{4}{(4\pi)^2\epsilon}[(3g^4+4\lambda^4a^2+\lambda^4b^2)\cos(p_1\wedge p_2+p_3\wedge p_4)+(3g^4+4\lambda^4ab+\lambda^4b^2)\cos(p_1\wedge p_3)\cos(p_2\wedge p_4)]. \tag{3}$$

The condition of one-loop renormalizability yields a system of two algebraic equations on a and b

$$3g^4 + 4\lambda^4a^2 + \lambda^4b^2 = ca, \tag{4}$$
$$3g^4 + 4\lambda^4ab + \lambda^4b^2 = cb, \tag{5}$$

where c is a constant. These equations are self-consistent only in the case $a = b$. Therefore the renormalizable potential for scalar electrodynamics has the form

$$V[\phi^*, \phi] = a\frac{\lambda^2}{2}(\{\phi^*,\,\phi\}_\star)^2, \tag{6}$$

where $\{f,\,g\}_\star = f \star g + g \star f$. Note that in contrast to the pure noncommutative complex scalar field theory [13] we do not have the solution $b = 0$.

Let us turn to an analysis of one-loop corrections to the 2-scalar-2-gluon vertex. The graphs that have non zero divergent parts in the $\alpha = 0$ gauge are presented in Figure 2.
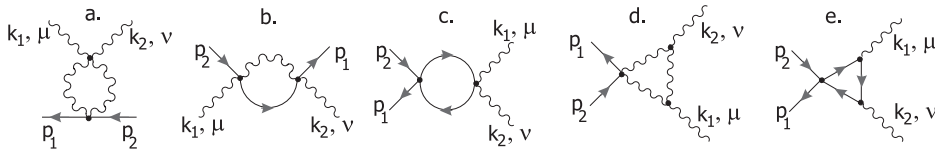

Figure 2: One-loop corrections to the 2-scalar-2-gluon vertex.

The sum of the divergent parts of these graphs is

$$\frac{12}{(4\pi)^2\epsilon}g^4\delta_{\mu\nu}[\cos(p_1 \wedge k_1 + p_2 \wedge k_2) - \cos(p_1 \wedge p_2)\cos(k_1 \wedge k_2)]. \tag{7}$$

Note that the graphs 2c and 2e do contain 4-point scalar vertex and there are terms depending on a and b. However the contributions of these graphs mutually cancel and the sum does not
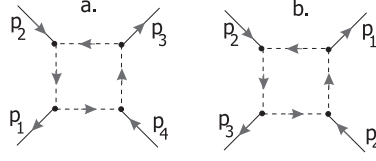
Figure 3: One-loop fermion corrections to the 4-point scalar vertex.

depend on a and b. Therefore, there are no new restrictions on these constants. We see that a counterterm requiring for a cancellation of (7) has just the same trigonometric structure as the initial 2-scalar-2-gluon vertex in the action, i.e. this vertex is one-loop renormalizable.

Thus the above analysis leads to the conclusion that noncommutative scalar electrodynamics (1) is one-loop renormalizable only if the scalar potential has the anticommutator form (6).

3 Noncommutative $\mathcal{N} = 2$ Super Yang-Mills Theory.

The action for the Euclidean noncommutative $\mathcal{N} = 2$ SUSY Yang-Mills theory reads

$$S = \int d^4x \left(-\tfrac{1}{4} F_{\mu\nu} \star F^{\mu\nu} + (\mathcal{D}_\mu \phi_-) \star (\mathcal{D}_\mu \phi_+) - i\chi^* \star \widehat{\mathcal{D}}\chi \right.$$
$$\left. -g\sqrt{2}\chi^* \star (R[\chi, \phi_+]_\star + L[\chi, \phi_-]_\star) - \tfrac{g^2}{2}([\phi_-, \phi_+]_\star)^2 \right), \tag{8}$$

where $\mathcal{D}_\mu = \partial_\mu - ig[A_\mu, \cdot]_\star$, $L, R = \tfrac{1}{2}(1 \pm \Gamma^5)$, $\phi_\pm$ are real scalar fields, χ is a complex four-component spinor[1]. The action (8) is a noncommutative generalization of Euclidean $\mathcal{N} = 2$ SYM theory [15]. A formulation of noncommutative $\mathcal{N} = 2$ supersymmetric theories in terms of superfields was given in [17].

Note that the scalar electrodynamics examined in the previous section can be considered as a bosonic part of $\mathcal{N} = 2$ NCSYM. The identification is evident: ϕ and ϕ^* corresponds to ϕ_+ and ϕ_-, respectively. Also we should replace λ by g and take $a = -b = -1$. The fact that fields $\phi_\pm$ are not complex conjugate does not affect the performed calculations. The Feynman rules for the bosonic part of the action (8) can be easily obtained from the Feynman rules for scalar electrodynamics (see Table 1) using the above mentioned identification. The Feynman rules for fermion fields are presented in Table 2.

Table 2: Feynman rules for fermion fields.

As in the case of scalar electrodynamics we start with the examination of the one-loop corrections to the 4-point scalar vertex. The graphs with fermion loops are presented in Figure 3. The divergencies coming from these graphs are

$$-\frac{32}{(4\pi)^2\epsilon} g^4 \cos(p_1 \wedge p_2 + p_3 \wedge p_4). \tag{9}$$

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[1]Note that $\mathcal{N} = 2$ SYM in Minkowski space contains a complex scalar field.
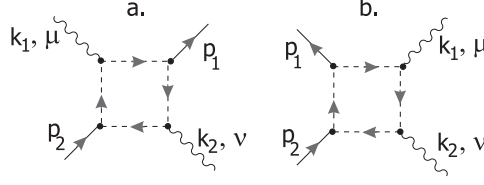
Figure 4: One-loop fermion corrections to the 2-scalar-2-gluon vertex.

Thus, taking into account the contribution (3) of the boson graphs we find that the boson and fermion divergencies mutually cancel. The similar result is valid for ordinary $\mathcal{N} = 2$ SUSY Yang-Mills theory where the 4-point scalar vertex is finite at one-loop [14].

Next we calculate one-loop corrections to the 2-scalar-2-gluon vertex. The graphs with fermion loops are presented in Figure 4. The sum of divergent parts of these graphs is

$$-\frac{16}{(4\pi)^2\epsilon}g^4\delta_{\mu\nu}[\cos(p_1 \wedge k_1 + p_2 \wedge k_2) - \cos(p_1 \wedge p_2)\cos(k_1 \wedge k_2)]. \tag{10}$$

Summing the contributions of the boson (7) and fermion (10) graphs we get

$$-\frac{4}{(4\pi)^2\epsilon}g^4\delta_{\mu\nu}[\cos(p_1 \wedge k_1 + p_2 \wedge k_2) - \cos(p_1 \wedge p_2)\cos(k_1 \wedge k_2)]. \tag{11}$$

Note that one-loop fermion corrections as well as boson ones restore the trigonometric structure of the initial vertex. So, we conclude that the noncommutative $\mathcal{N} = 2$ $D = 4$ SYM with the action (8) is one loop renormalizable.

4   Infrared Behaviour.

The finite parts of Feynman graphs contain the terms, which have nonanalytic behaviour in ξ and in external momenta. There is a nontrivial mixing of UV and IR divergencies (see [18, 10] for details). It is straightforward to show that a type of the UV divergency of the integral $J_{UV}(\xi p) = \int f(k, p)dk$ coincides with a type of the IR behaviour of the integral $J_{IR}(\xi p) = \int e^{i\xi k\theta p}f(k, p)dk$. For example, the logarithmically divergent integral J_{UV} yields a logarithmic singularity $\log(\xi|\theta p|)$ in J_{IR}, the quadratically divergent integral J_{UV} yields a quadratic singularity $(\xi|\theta p|)^{-2}$ in J_{IR}, and so on. One has to care only about the poles, since near the origin the logarithm is an integrable function. IR poles appear in the corrections to propagators and can produce IR divergencies in multi-loop graphs.

The one-loop correction to the gauge propagator has been computed in [16] and it has only logarithmic IR singularity. We examine one-loop corrections to the scalar field propagator in the case of noncommutative scalar electrodynamics (1) and the noncommutative $\mathcal{N} = 2$ NC SYM theory (8). We do not analyze one-loop corrections to the fermion propagator since they have only logarithmic UV divergencies.

In the case of scalar electrodynamics all corrections to the scalar field propagator are presented in Figure 5a,b,c. For $a = b$ the sum of the divergent parts is:

$$\frac{6g^2}{(2\pi)^4}\int \frac{d^4k}{k^2}\left[1 - \frac{a\lambda^2}{g^2} - \left(1 + \frac{a\lambda^2}{3g^2}\right)\cos(2k \wedge p)\right] - \frac{8g^2}{(2\pi)^4}\int d^4k\frac{1 - \cos(2k \wedge p)}{k^2(p + k)^2}\left[p^2 - \frac{(pk)^2}{k^2}\right] \tag{12}$$
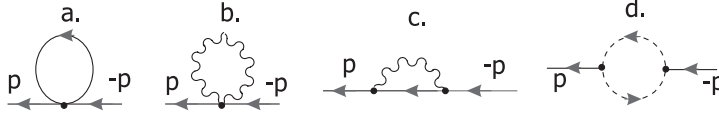
Figure 5: One-loop corrections to the scalar field propagator

where p is an external momentum. A quadratic UV divergence in (12) is removed by a mass renormalization. To remove IR poles one has to impose the condition $\lambda^2 a = -3g^2$.

Next we compute one-loop correction to the scalar field propagator in the $\mathcal{N} = 2\ D = 4$ NC SYM (8). We have one more graph presented in Figure 5d. Summing all contributions we have

$$- 4g^2 \int \frac{d^4k}{(2\pi)^4} \frac{1 - \cos(2k \wedge p)}{k^2(p+k)^2} \left[p^2 + 2\frac{(pk)^2}{k^2} \right] \tag{13}$$

It is remarkable, that all quadratically divergent integrals vanish. Therefore, only the term with an integrable logarithmic IR singularity appears after integration.

Acknowledgments

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